A New Approach to Covering

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Abstract

For finite coverings in euclidean d-space E^d we introduce a parametric density function. Here the parameter controls the influence of the boundary of the covered region to the density.

This definition gives an new approach to covering which is similar to the approach for packing in [BHW]. In this way we obtain a unified theory for finite and infinite covering and generalize similar results for $d = 2$ developed by various authors since 1950 to all dimensions.

1 Introduction

Infinite packing and covering, in particular lattice packing and covering of euclidean d-space E^d by spheres or other convex bodies is an important and well established part of mathematics. But all packings and coverings in real world are finite. So a theory of finite packing and covering is desirable. This theory should of course comprise the infinite theory as a limiting case. Starting from the usual definition of packing and covering density (cf. e.g. [GL] or [R]), one is led to packings and coverings of a convex bodies with respect to another "large" convex body. These restricted packings and coverings (in the case of packings usually called "bin packings") have been studied by various authors (cf. [GW] OR [K]).

In many situations there is no natural large body associated to the packing or covering. A natural way of denotation of this type of packing or covering is "free" packing or covering. A way to measure the quality of these free packings and coverings is to measure the volume or other functional of some associated convex body.

About 1950 Rogers, Bambah, L. Fejes Tóth, Hadwiger, Zassenhaus and later Groemer, Oler, Woods, Witsenhausen, Folkman, Graham, G. Wegner et al. introduced such free packings and coverings (cf. [GW]). These authors showed for $d = 2$ various relations between classical infinite packing and covering and free packing and covering. But the work of these authors was restricted to the plane.

A first step to E^d was by L. Fejes Tóth's famous sausage conjecture that for $d \geq 5$ linear arrangements of balls have minimal volume of the convex hull under all arrangements of the same cardinality. By its nature the work stimulated by the sausage conjecture (cf. again [GW]) had no connection to the theory of infinite packing and covering. For packing in [BHW 1], see also [BHW 2] a parametric density function was introduced, such that for small values of the parameter sausage packings are optimal while for large values of the parameter the classical density of infinite packings was obtained as limit of the finite optimal densities. In fact the method used there was strong enough to prove the sausage conjecture for almost all dimensions. In this paper we show that a suitable generalization of the parametric density gives a theory of free finite covering which comprises the classical definition of covering as a limiting case for suitable parameters. Altogether we obtain a relation between free finite packing and free finite covering which is analogous to the infinite case:

"Small" values of the parameter lead to lowdimensional packings and highdimensional coverings while "large" parameters lead to highdimensional packings and lowdimensional coverings (though sausages are never optimal coverings).

2 Definitions

Let $\mathcal{K}^d,\ d \geq 2$ denote the set of convex bodies K with volume $V(K) > 0$ in euclidean d-space E^d. For $K \in \mathcal{K}^d$ and $c_1, \dots, c_n \in E^d$ let $K_i = K + c_i,\ i = 1, \dots, n$ and $C_n = \{c_1, \dots, c_n\}$. We call C_n an arrangement and define the density function of this arrangement by

$$\Theta(K, C_n, \rho) = nV(K)/V(\mathrm{conv}C_n + \rho K) \tag{1}$$

where $\rho \in \mathbf{R}$ is a parameter. For $\rho \geq 0$ $\mathrm{conv}C_n + \rho K$ denotes the usual Minkowski sum and for $\rho < 0$ the Minkowski difference (cf. [S] p. 133–137 or [GW] p. 877), sometimes also called the inner parallel body (cf. [S] p. 134). The Minkowski difference has various applications in convex geometry (cf. [S] p. 137 and 350) and is defined by

$$\mathrm{conv}C_n + \rho K = \{x \in E^d \mid x + |\rho| K \subset \mathrm{conv}C_n\} \quad \text{for } \rho < 0. \tag{2}$$

In particular we have, if $\mathrm{conv}C_n - K \neq \emptyset$: $(\mathrm{conv}C_n - K) + K \subset \mathrm{conv}C_n$. The minimal ρ, for which $\mathrm{conv}C_n + \rho K \neq \emptyset$, is called the inradius $\rho(\mathrm{conv}C_n, K)$ of C_n with respect to K. In the following we always assume tacitly $\rho > \rho(\mathrm{conv}C_n, K)$. Finally, if $\mathrm{conv}C_n + \rho K = \emptyset$ we set $\Theta(K, C_n, \rho) = \infty$. By our definition clearly $\Theta(K, C_n, \rho) > 0$ for all convex bodies, arrangements and parameters.

While the definition of density pertains to arbitrary arrangements and could in fact easily be generalized to arrangements of congruent copies of K, we are here interested in more specific arrangements, so we define:

a) If $\mathrm{int}(K_i \cap K_j) = \emptyset$ for $i \neq j$, then C_n is a *packing.*

b) If $\mathrm{conv}C_n \subset C_n + K$, then C_n is a *covering.*

c) If C_n is a packing and a covering, then C_n is a *tiling.*

Of course one is interested in optimal arrangements, in particular densest packings and thinnest (most economical) coverings. This leads for given $K \in \mathcal{K}^d,\ n \in \mathbf{N}$ and $\rho \in \mathbf{R}$ to:

$$\delta(K, n, \rho) = \sup\{\Theta(K, C_n, \rho) \mid C_n \text{ packing}\} \tag{3}$$

$$\vartheta(K, n, \rho) = \inf\{\Theta(K, C_n, \rho) \mid C_n \text{ covering}\} \tag{4}$$

$$\delta(K, \rho) = \limsup_{n \to \infty} \delta(K, n, \rho) \tag{5}$$

$$\vartheta(K, \rho) = \liminf_{n \to \infty} \vartheta(K, n, \rho) \tag{6}$$

As $\delta(K, n, \rho) = \infty$ for $\rho \leq 0$ we consider $\delta(K, n, \rho)$, $\delta(K, \rho)$ only for $\rho > 0$.

If an arrangement C_n is of the form $C_n = (\mathrm{conv}C_n) \cap L$ for some lattice $L \subset E^d$, then we call C_n a lattice arrangement. For lattice packings and coverings we define the lattice densities: $\delta_L(K, n, \rho)$, $\vartheta_L(K, n, \rho)$, $\delta_L(K, \rho)$, $\vartheta_L(K, n)$ analogously to (3) to (6).

Finally in the theory of finite packings sausages take an important part. We say that the arrangement C_n is a sausage, if it is of the form $C_n = \{iu | u \in E^d,\ |u| \neq 0,\ i = 1, \dots, n\}$. We observe that for small $|u|$ C_n yields a covering, while for large $|u|$ it yields a packing.

3 Results

While we treat coverings in this note we state some of the results on packings as well to show that for finite packing and covering holds an analogous complementarity as in the infinite case.

The density has the following simple properties:

Proposition 1 *(a)* $\vartheta(K,n,\rho) > 0$ *and* $\vartheta(K,\rho) > 0$,

(b) $\vartheta(K,n,\rho) \geq 1$ *and* $\vartheta(K,\rho) \geq 1$ *for* $\rho \leq 0$

(c) $\vartheta(K,n,\rho)$ *and* $\vartheta(K,\rho)$ *are invariant under affine mappings of* E^d.

The following result links for all $d \geq 2$ finite coverings and the classical density $\vartheta(K)$ of thinnest coverings of E^d by translates of K (cf. [R] or [FK] or [GW]).

Theorem 2 *For* $K \in \mathcal{K}^d$, $\rho \leq -(d+1)$ *and for* $K \in \mathcal{K}^d$, $K = -K$ $\rho \leq -2$ *holds*

$$\vartheta(K,n,\rho) \geq \vartheta(K),\ n \in \mathbf{N} \qquad \vartheta(K,\rho) = \vartheta(K).$$

Theorem 3 (BHW 1) *For* $K \in \mathcal{K}^d$, $\rho \geq d+1$ *and for* $K \in \mathcal{K}^d$, $K = -K$, $\rho \geq 2$ *holds*

$$\delta(K,n,\rho) \leq \delta(K),\ n \in \mathbf{N} \qquad \delta(K,\rho) = \delta(K).$$

Theorem 4 *For* $K \in \mathcal{K}^d$, $\rho \leq -d$ *and for* $K \in \mathcal{K}^d$, $K = -K$, $\rho \leq -1$ *holds*

$$\vartheta_L(K,n,\rho) \geq \vartheta_L(K),\ n \in \mathbf{N} \qquad \vartheta_L(K,\rho) = \vartheta_L(K).$$

Theorem 5 (H) *For* $K \in \mathcal{K}^d$, $\rho \geq (3/2)(d+1)$, *for* $K \in \mathcal{K}^d$, $K = -K$, $\rho \geq 3$ *and for* $K = B^d$, $\rho \geq \sqrt{21}/2$ *holds*

$$\delta_L(K,n,\rho) \leq \delta_L(K),\ n \in \mathbf{N} \qquad \delta_L(K,\rho) = \delta_L(K).$$

Remark: Most of the bounds for ρ especially in Theorems 2, 4 seem to be far best from possible.

The theorems above show that parametric finite covering (parametric finite packing) gives for small ρ (large ρ) a new approach to infinite covering (infinite packing). Thus small parameters for covering correspond to large parameters for packing. On the other hand there appears to be no straightforward correspondence between large parameters for covering and small parameters for packing. The role of sausages in the theory of finite packings is demonstrated by the following theorem (cf. [BHW 1, BHW 2]) which we state here only in a qualitative way.

Theorem 6 (BHW 1) *For every dimension* d *there is a constant* $c_d > 0$ *such that for all* $K \in K^d$, *all* $n \in \mathbf{N}$ *and all* $\rho \leq c_d$, $\delta(K,n,\rho) = \delta(K,S_n,\rho)$ *for a suitable sausage* S_n.

The following propostion shows that the situatation for coverings is different:

Proposition 7 *For all* n,d,ρ *with* $\rho > 0$, $d \geq 2$, $n \geq 3$ $\quad \vartheta(B^d,n,\rho) > \vartheta(S_n,\rho)$.

The lack of a counterpart of Theorem 6 leaves a certain gap in the theory of finite covering in the following sense: By Theorem 6 we have a tight upper bound for $\delta(K,C_n,\rho)$ for all sufficiently small ρ. This corresponds to a lower bound for $\vartheta(K,C_n,\rho)$ for large ρ. The following Theorem partially closes this gap:

Theorem 8

$$\vartheta(K,n,\rho) \geq \left\{ d^d(2^d(1+\rho)^{d-1} + \frac{1}{n}\rho^d) \right\}^{-1}$$

Corollary 9

$$2(2d)^{-1}(\rho+1)^{-(d-1)} \leq \vartheta(K,\rho) \leq e(\rho+1)^{-(d-1)} \quad \rho \geq 0$$

We close with a look at two-dimensional covering. For two-dimensional parametric packing densities there holds the following alternative which in a certain sense solves the problem for centrally symmetric bodies: Up to a constant ρ_0 depending only on the body $\delta(K,n,\rho)$ is given by a sausage and for all $\rho > \rho_0$ the density is less than the density of the densest infinite packing (cf. [BHW 1]). This results depends on a result of Oler [O]. There is a counterpart to Oler's Theorem by Bambah, Rogers and Zassenhaus. Though their result is optimal it is of little help in our context, as their condition is of a combinatorial rather than metrical nature.

Here we can only give a metrical counterpart to Oler's theorem for circles, which is a counterpart of a theorem of Groemer [G] for packing and as such of some interest in its own. From this we derive a theorem for finite covering by circles which is at least asymtotically (with respect to n) sharp. To state the theorem we define for a configuration C_n the number $H(C_n)$ of its "boundary points" by $H(C_n) = \mathrm{card}(C_n \cap \mathrm{bd}\,\mathrm{conv}C_n)$ and we use the constants

$$\alpha = \frac{\sqrt{2}}{4} \frac{\left(\left(4+\sqrt{17}\right)^{2/3} - 1\right)^{3/2}}{\sqrt{4+\sqrt{17}}} = 0.6578.., \quad \beta = \frac{1}{2\sqrt{1-\alpha^2}} - \frac{4\sqrt{3}\alpha}{9} = 0.1574..,$$

$$\gamma = 4 - \frac{8\alpha}{3\sqrt{3}} - (2\alpha+2)\beta = 2.465\ldots$$

Theorem 10 *Let $C_n \subset E^2$ be a covering set. Then*

i.

$$n \geq \frac{2}{3\sqrt{3}} V_2(\mathrm{conv}C_n) + 1/2H(C_n) + 1.$$

There are infinitely many n and C_n such that equality holds in (10).

ii.

$$n \geq \frac{2}{3\sqrt{3}} V_2(\mathrm{conv}C_n) + \beta V_1(\mathrm{conv}C_n) + 1$$

Further for every $n \geq 5$ there is a C_n such that $n \leq \frac{2}{3\sqrt{3}} V_2(\mathrm{conv}C_n) + \beta V_1(\mathrm{conv}C_n) + \gamma$.

Remarks:

1. Of course the first part of the theorem is just a special case of the Theorem of Bambah, Rogers and Zassenhaus. We have included a proof as most of our proof is the same as the second part of the theorem and probably somewhat simpler than the general case.

2. The examples in the second part show that the coefficients of V_2, V_1 cannot be improved. A single point shows that the constant 1 cannot be improved. The examples have the shape of bones, which indicates that probably the role of sausages in finite packing is taken by bones in covering (cf. [GW]).

The immediate consequence of Theorem 10 for finite covering is

Corollary 11 *For all $\rho \leq \frac{3\sqrt{3}}{4}\beta$ $\vartheta(B^2,n,\rho) > \vartheta(B^2)$ and for all $\rho \geq \frac{3\sqrt{3}}{4}\beta + \epsilon_n$ $\vartheta(B^2,n,\rho) < \vartheta(B^2)$ for some sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$.*

4 Proof of the statements for general d

The proof of Theorem 2 is based on the following idea: assume that $\mathrm{conv}C_n + \rho K$ is a finite covering with $\vartheta(K, n, \rho) < \vartheta(K)$. Then a packing lattice Λ of $\mathrm{conv}C_n + \rho K$ with elementary cell Z is chosen. For every $x \in Z$ the lattice packing $L(\mathrm{conv}C_n + \rho K + x) = \{(\mathrm{conv}C_n + \rho K + x) + g \mid g \in \Lambda\}$ is superposed on a densest infinite covering $\{K + a \mid a \in C(K)\}$ with density $\vartheta(K)$. Further all $K + a$, $a \in C(K)$ which meet $L(\mathrm{conv}C_n + \rho K + x)$ are deleted, where $\rho \leq -2$ and (2) guarantee that we obtain again a covering of E^d.

A standard averaging argument with respect to x shows the existence of an infinite covering of translates of K with density less than $\vartheta(K)$ which contradicts the definition of $\vartheta(K)$. Hence $\vartheta(K, n, \rho) \geq \vartheta(K)$. The proof gives a careful analysis of this idea:

Assume there exist $K \in \mathcal{K}^d$ and $\rho \leq -2$ satisfying the assumption and an integer n with $\vartheta(K, n, \rho) < \vartheta(K)$. Then there is an $C_n \in \mathcal{C}_n$ and an $\epsilon > 0$ with

$$\vartheta(K) = \frac{nV(K) + \epsilon\vartheta(K)}{V(\mathrm{conv}C_n + \rho K)}. \tag{7}$$

Let Λ be a packing lattice of $\mathrm{conv}C_n + \rho K$. We may assume that $\mathrm{conv}C_n + \rho K$ is contained in a fixed elementary cell Z of Λ. From (7) follows easily

$$\left(1 - \frac{V(\mathrm{conv}C_n + \rho K)}{\det(\Lambda)}\right)\vartheta(K)\frac{\det\Lambda}{\det(\Lambda) - \epsilon} + \frac{nV(K)}{\det(\Lambda) - \epsilon} = \vartheta(K). \tag{8}$$

Now for $\lambda > 0$ let $W_\lambda \in \mathcal{K}_0^d$ be the cube of edge length 2λ. Apparently there is a constant μ depending only on Z such that for every $\lambda > 0$ there is a subset $L_\lambda \subset \Lambda$ such that $W_\lambda + Z \subset L_\lambda + Z$ and $L_\lambda + 2Z \subset W_{\lambda+\mu}$.

By the definition of $\vartheta(K)$ for every $\lambda > 0$ there exists a set $C_{m(\lambda)} \in \mathcal{C}_{m(\lambda)}(K)$ such that $C_{m(\lambda)} + K \supset W_\lambda$ and

$$\lim_{\lambda\to\infty} \frac{m(\lambda)V(K)}{V(W_\lambda}) = \vartheta(K).$$

Obviously $\lim_{\lambda\to\infty} V(W_{\lambda+\mu})/V(W_\lambda) = 1$, so there exist a $\zeta > 0$ and a set $C_m(\zeta) \in \mathcal{C}_{m(\zeta)}(K)$ with $C_{m(\zeta)} + K \supset W_\zeta$ such that

$$\vartheta(K)\frac{\det\Lambda}{\det\Lambda - \epsilon} > \frac{m(\zeta)V(K)}{V(W_{\zeta+\mu})} \quad \text{and} \quad \frac{nV(K)}{\det(\Lambda) - \epsilon} > \frac{nV(K)\mathrm{card}(L_\zeta)}{V(W_{\zeta+\mu})} \tag{9}$$

For every $x \in Z$ we construct a finite covering $C_{n(x)} \in \mathcal{C}_{n(x)}$ – for a suitable $n(x) \in \mathbf{N}$ – with $C_{n(x)} + K \supset W_{\zeta+\mu}$ in the following way:

$$C_{n(x)} = \{x + L_\zeta + C_n\} \cup \{y \in C_{m(\zeta)} \mid y \notin x + L_\zeta + (\mathrm{conv}C_n + \rho K)\}.$$

Now (?) and $\rho \leq -2$ guarantee that $C_{n(x)}$ is a covering. While it is difficult to determine the cardinality $n(x)$ of $C_{n(x)}$ for fixed x it is easy to calculate $\int_{x\in Z} n(x)dx$ as follows:

For every $y \in C_{m(\zeta)}$ let $\chi_y(x) = 1$ for $y \notin x + L_\zeta + \mathrm{conv}C_n + \rho K$ and $\chi_y(x) = 0$ else. Then

$$\begin{aligned}\int_{x\in Z} n(x)dx &= \int_{x\in Z}\left(n\mathrm{card}(L_\zeta) + \sum_{y\in C_{m(\zeta)}} \chi_y(x)\right)dx \\ &= n\det(\Lambda)\mathrm{card}(L_\zeta) + m(\zeta)\left(\det(\Lambda) - V(\mathrm{conv}C_n + \rho K)\right).\end{aligned}$$

So there is a $z \in Z$ with

$$n(z) \leq m(\zeta)\left(1 - \frac{V(\text{conv}C_n + \rho K)}{\det(\Lambda)}\right) + n\text{card}(L_\zeta)$$

or

$$\frac{n(z)V(K)}{V(W_{\zeta+\mu})} \leq \frac{m(\zeta)V(K)}{V(W_{\zeta+\mu}}\left(1 - \frac{V(\text{conv}C_n + \rho K)}{\det(\Lambda)}\right) + \frac{nV(K)\text{card}(L_\zeta)}{V(W_{\zeta+\mu}}$$

From (8) and (9) follows

$$\frac{n(z)V(K)}{V(W_{\zeta+\mu})} < \vartheta(K). \tag{10}$$

But $C_{n(z)} + K \supset W_{\zeta+\mu}$ and thus (10) contradicts the definition of $\vartheta(K)$.

Proof of Theorem 4: First we show the assertion for 0-symmetric convex bodies $K \in \mathcal{K}^d$. To this end let f_K denote the distance function of K and for a d-dimensional lattice $L \subset E^d$ let

$$H(K,L) := \{x \in E^d : f_K(x) \leq f_K(x-a),\ a \in L\}$$

be the honeycomb (Wabenzelle) of K with respect to L (cf. [Gr], [GL]). Further we denote by $\mu(K,L)$ the inhomogeneous minimum of K for the lattice L (cf. [GL]). The inhomogeneous mimimum is closely related to the honeycomb. Namely, with respect to the metric given by a distance function f_K, $\mu(K,L)$ may be considered as the circumradius of $H(K,L)$, that is

$$\mu(K,L) = \min\{\rho \in R^{>0} : H(K,L) \subset \rho \cdot K\}. \tag{11}$$

Now let $K \in \mathcal{K}^d$, $K = -K$, L be a covering lattice of K and let $C_n = (\text{conv}C_n) \cap L$ be a finite lattice arrangement with respect to K. We claim

$$V(\text{conv}(C_n) - \mu(K,L)K) \leq n \cdot \det(L). \tag{12}$$

First we assume that K is strictly convex, i.e. for $x, y \in K$, and $\lambda \in (0,1)$ the points $\lambda x + (1-\lambda)y$ are inner points of K. In this case $H(K,L)$ is a well-defined starbody which generates a lattice tiling of the space (cf. [H]). Hence

$$V(H(K,L)) = \det(L)$$

and for (12) it suffices to prove

$$\text{conv}C_n - \mu(K,L)K \subset C_n + H(K,L).$$

Let $x \notin C_n + H(K,L)$. Then there exists a $z \in L$ with $z \notin \text{conv}C_n$ and $x \in z + H(K,L)$, which implies by the 0-symmetry of $H(K,L)$: $z \in x + H(K,L)$ and thus $z \in x + \mu(K,L)K$ (cf. (11)). Since $z \notin \text{conv}C_n$ we get $x \notin \text{conv}C_n - \mu(K,L)K$.

Now let K be an arbitray 0-symmetric convex body. There exists a sequence $\{K_i\}$ of 0-symmetric strictly convex bodies with $K \subset K_i$ and $K_i \to K$ as i tends to infinity (cf. [BF, pp. 35]). We may assume that 0 is an inner point of $\text{conv}(C_n) - \mu(K,L)K$. Then it is easy to see that for suitable numbers $\epsilon_i > 0$, $\epsilon_i \to 0$, $i \to \infty$, and sufficiently large i holds

$$(1-\epsilon_i) \cdot (\text{conv}C_n - \mu(K,L)K) + \mu(K_i,L)K_i \subset \text{conv}C_n.$$

Hence we find

$$\begin{aligned} V(\text{conv}C_n - \mu(K,L)K) &\leq (1/(1-\epsilon_i))^d V(\text{conv}C_n - \mu(K_i,L)K_i) \\ &\leq (1/(1-\epsilon_i))^d n \cdot \det(L). \end{aligned}$$

Since L is a covering lattice we have $\mu(K,L) \leq 1$ and so by (12)

$$V(\text{conv}C_n - K) \leq n \cdot \det(L).$$

By the definition of $\vartheta_L(K)$ this yields $\vartheta_L(K) \cdot V(\text{conv}C_n - K) \leq n \cdot V(K)$ for any finite lattice covering arrangement C_n. Thus

$$\vartheta_L(K) \leq \vartheta_L(K, n, -1). \tag{13}$$

On account of (???) we obtain $\vartheta_L(K,-1) = \vartheta_L(K)$.

Finally let $K \in \mathcal{K}^d$. Since $\vartheta(K,n,\rho)$, $\vartheta(K,\rho)$ are invarinat with respect to affine transformations (cf. Proposition 1) we may assume that the centroid of K is the origin. Hence (cf. [BF, p. 34, p. 73])

$$K \subset \frac{d}{d+1}(K-K) \subset dK$$

So if L is a lattice covering of K then it is also for the 0-symmetric convex body $(d/(d+1)) \cdot (K-K)$. Now let C_n as above. It follows

$$\begin{aligned}
\frac{nV(K)}{V(\text{conv}C_n - dK)} &= \frac{V(K)}{V(d/(d+1)(K-K))} \cdot \frac{nV((d/(d+1)(K-K))}{V(\text{conv}C_n - dK)} \\
&\geq \frac{V(K)}{V(d/(d+1)(K-K))} \cdot \frac{nV((d/(d+1)(K-K))}{V(\text{conv}C_n - d/(d+1)(K-K))} \\
&\geq \frac{V(K)}{V(d/(d+1)(K-K))} \cdot \frac{V(d/(d+1)(K-K))}{\det(L)} = \frac{V(K)}{\det(L)} \geq \vartheta_L(K).
\end{aligned}$$

Proof of Proposition 7: As $\Theta(B^d, S_n, \rho) = \infty$ for $\rho \leq 0$ we have to find for every n, d and $\rho > 0$ a covering C_n such that $V(\text{conv}(C_n) + \rho B_d) > V(\text{conv}(S_n) + \rho B_d)$. For simplicity this is only done for odd $n = 2k+1$ but the examples are easily modified for even n.

We have $V(\text{conv}(S_n) + \rho B^d) = 2(n-1)\kappa_{d-1}\rho^{d-1} + \kappa_d \rho^d$. For any covering C_n we have by Steiner's formula for $\rho \geq 0$

$$V(\text{conv}(C_n) + \rho B^d) = V_0(\text{conv}C_n)\kappa_d\rho^d + V_1(\text{conv}C_n)\kappa_1\rho^{d-1} + \ldots + V_d(\text{conv}C_n),$$

where V_i denotes the i-th intrinsic volume (cf. e.g. [S] p. ?). Specifically for a 2-dimensional set $\text{conv}C_n$, $V_2(\text{conv}C_n)$ is the area, $V_1(\text{conv}C_n)$ is half the perimeter, $V_0(\text{conv}C_n) = 1$ and $V_i(\text{conv}C_n) = 0$ for $i > 0$.

For all n, d, ρ the examples C_n consist of points of suitably chosen triangles: Let $\epsilon \in \mathbf{R}$ such that $0 < \epsilon < 1$ and $s = 2\sqrt{1-\epsilon^2}$. The points of C_n have the form $x^i = (is, i\epsilon/k, 0, \ldots, 0)$, $i = 0, \ldots, k$, $x^i = (is, (2 - i/k)\epsilon, 0, \ldots, 0)$, $i = k+1, \ldots, 2k$. It is easily checked that for all ϵ, n the C_n form a covering.

We find

$$\begin{aligned}
V((\text{conv}C_n) + \rho B^d) &= \kappa_d\rho^d + V_1(\text{conv}C_n)\kappa_{d-1}\rho^{d-1} + V_2(\text{conv}C_n)\kappa_{d-2}\rho^{d-2} \\
&\geq \kappa_d\rho^d + 4k\sqrt{1-\epsilon^2}\kappa_{d-1}\rho^{d-1} + 2k\epsilon\sqrt{1-\epsilon^2}\kappa_{d-2}\rho^{d-2} \\
&> \kappa_d\rho^d + 4k\kappa_{d-1}\rho^{d-1} = V((\text{conv}S_n) + B^d)
\end{aligned}$$

if ϵ is chosen appropriately.

Proof of Theorem 8: Let $K \in \mathcal{K}^d$. After a suitable affine transformation we have by John's theorem (cf. [???]) $\frac{1}{d}B^d \subset K \subset B^d$. So for given $n \in \mathbf{N}$ and $\rho \geq 0$ we obtain by STEINER's formula:

$$\vartheta(K, C_n, \rho) = nV(K)/V(\text{conv}C_n + \rho K) \geq n\kappa_d / \left(d^d \sum_{i=0}^{d} V_{d-i}(\text{conv}C_n)\kappa_i \rho^i \right).$$

With $V_{d-i}(\text{conv}C_n) \leq nV_(d-i)(B^d), \quad I = 0, \dots, d$ follows

$$\vartheta(K, C_n, \rho) \geq \left\{ d^d \left(\sum_{i=0}^{d-1} \binom{d}{d-i} \frac{\kappa_i}{\kappa_{d-i}} \rho^i + \frac{1}{n}\rho^d \right) \right\}^{-1}.$$

Further simplifications ????? qed

Proof of the corollary: ????

5 Proof of the twodimensional results

We need some more notation for our proofs: We write $B(x)$ rather than $B^2 + x$ and for $x_1, \dots, x_n$ we write $[x_1, x_2]$ for the line segment joining x_1 and x_2 and $[x_1, \dots, x_n]$ for the polygon with vertices $x_1, \dots, x_n$ (in this order). While the polygons considered here will generally not be convex they will always be topological disks.

We begin our proof by stating some simple general lemmas:

Lemma 12 *Let $x, y, z \in E^2$ such that $\Delta = [x, y, z] \subset B(x) \cup B(y) \cup B(z)$. Then $V(\Delta) \leq 3\sqrt{3}/4$ and equality holds for an equilateral triangle with edge-length $\sqrt{3}$.*

Proof: Immediate consequence of the densest lattice packing of circular discs.

Lemma 13 *Let $Q = [a, b, c, d]$ be a quadrangle, such that $[b, c]$, $[a, d]$ are parallel and $[a, d]$, $[b, c]$ are perpendicular on $[c, d]$. Further let $Q \subset B(a) \cup B(b)$ and $|c - d| = \beta$. Then $V(Q) \leq V(Q')$ where Q' is a quadrangle with the same properties as Q and, additionally, $|a - d| = |b - d|$ and $|a - (c + d)/2| = |b - (c + d)/2| = 1$.*

Proof: We may assume $|a - d| = \alpha,\ |b - c| = \alpha + \delta,\ \delta \geq 0,\ \beta = \sqrt{1 - \alpha^2} + \sqrt{1 - (\alpha + \delta)^2}$. Then it is clearly sufficient to show, that the quadrangle with

$$|a - d| = \alpha + \delta/2,\ |b - c| = \alpha + \delta/2,\ \delta \geq 0,\ \beta = \sqrt{1 - \alpha^2} + \sqrt{1 - (\alpha + \delta)^2}.$$

is covered by $B(a) \cup B(b)$. This follows from

$$2\sqrt{1 - (\alpha + \delta/2)^2} - \sqrt{1 - (\alpha + \delta)^2} - \sqrt{1 - \alpha^2} \geq 0$$

which is easily checked by differentiation with respect to δ. qed

Now let C_n be a finite covering. As ROGERS] [R] for infinite coverings by balls in arbitrary dimension we construct the Delaunay triangulation of convC_n: For a in C_n we note by $D(a)$ its Voronoi cell $D(a) = \{x \in E^2 \mid |x - a| \leq |x - b| \text{ for all } b \in C_n\}$. We join $a, b \in C_n$ if $D(a)$ and $D(b)$ have a common edge. If the resulting tesselation of convC_n is not a triangulation we take an arbitrary triangulation of every n-gon for $n \geq 4$ which introduces no additional vertices (a n-gon is thus subdivided in $n-2$ triangles). We observe that for every triangle $[a,b,c]$ of the Delauney triangulation $D(a), D(b), D(c)$ have a common vertex.

We say that $[a,b]$ is edge of convC_n if $[a,b]$ is line segment of the triangulation that is contained in only one triangle. An edge $[a,b]$ is called a long edge, if $[a,b]\cap D(c)\neq\emptyset$ for a $c\in C_n\setminus\{a,b\}$. For each long edge $[a,b]$ we construct a polygon $Q(a,b)=[d_1,\dots,d_n]$ in the following way: Let $d_1=a$, $d_n=b$ and let the other d_n be determined by the following properties: $D(d_i)\cap D(d_i-1)$ is a line segment and $D(d_i)\cap[a,b]\neq\emptyset$. For every vertex d_i of $Q[a,b]$ we denote by $p(d_i)$ the orthogonal projection of d_i on aff$[a,b]$.

Lemma 14 *$Q(a,b)$ is a (generally) nonconvex polygon, such that for $i=1,\dots,n-1$ $[d_i,d_{i+1}]$ is an edge of the triangulation and $[d_i,d_{i+1},p(d_{i+1}),p(d_i)]\subset B(d_i+1)\cup B(d_i)$.*

Proof: By construction $[d_i,d_{i+1}]$ is an edge of the triangulation. Further $[a,b]\subset\bigcup_{i=1}^n B(d_i)$. Now the lemma follows from easy elementary considerations. qed

Lemma 15 *Let $[a_1,a_2,a_3]$ be a triangle of the triangulation such that $[a_1,a_2,a_3]\not\subset B(a_1)\cup B(a_2)\cup B(a_3)$. Then there is a long edge $[p,q]$ of P such that $[a_1,a_2,a_3]\subset Q(p,q)$.*

Proof: Let $b\in[a_1,a_2,a_3]\setminus(B(a_1)\cup B(a_2)\cup B(a_3))$ and $c=D(a_1)\cap D(a_2)\cap D(a_3)$. We have by our construction $1<\min\{|b-a_i|\}\leq|c-a_i|=\min\{|x-c|\mid x\in C_n\}$. As C_n is a covering set, this gives $c\notin$ convC_n. This gives bd conv$C_n\cup D(a_i)\neq\emptyset$ for $i=1,2,3$. As we have again by construction that conv$\{a_1,a_2,a_3,y\}$ contains no element of C_n in its interior, the sets $D(a_i)\cup$ bd convC_n, $i=1,2,3$ must be contained in one long edge of convC_n. qed

Proof of Theorem 10: If T denotes the number of triangles in the triangulation we have $n=T/2+H/2+1$. By lemmas 15, 12 we have for every triangle Δ of the triangulation which is not contained in any $Q(a,b)$ $\frac{2}{3\sqrt{3}}A(\Delta)\leq 1/2$. Further we have that for a long edge $[a,b]$ of P that $Q(a,b)=[d_1,\dots,d_n]$ contains $n-2$ triangles of the triangulation. For a fixed $Q(a,b)$ we define $Q_i=[d_i,d_{i+1},p(d_{i+1}),p(d_i)]$. Thus Q_1 and Q_{n-1} are triangles and Q_i, $i=2,\dots,n-2$ are quadrangles.

To prove (i) it remains to show

$$\frac{2}{3\sqrt{3}}A(Q_i)\leq 1/2,\ i=2,\dots,n-2;\quad \frac{2}{3\sqrt{3}}A(Q_i)\leq 1/4,\ i=1,n-1. \tag{14}$$

For $i=2,\dots,n-2$ we may assume

$$|d_i-p(d_i)|=|d_{i+1}-p(d_{i+1})|=\alpha,\quad |p(d_i)-p(d_{i+1})|=2\sqrt{1-\alpha^2}$$

by lemma 13. Thus $A(Q_i)=2\alpha\sqrt{1-\alpha^2}$. $A(Q_i)$ becomes maximal for $\alpha=\sqrt{2}/2$ and in this case we have $A(Q_i)=1$ and inequality (14) is clearly satisfied. Q_1 is a rectangular triangle with right angle at $p(d_2)$ and $|d_2-p(d_2)|=\alpha\leq 1$. Thus $A(Q_1)\leq\alpha(1+\sqrt{1-\alpha^2})/2$. Differentiation with respect to α gives immediately that $A(Q_1)$ becomes maximal for $\alpha=\sqrt{3}/2$ and $A(Q_1)\leq 3\sqrt{3}/8$. Thus (14) is again satisfied. Q_{n-1} is treated in the same way. Suitable portions of the thinnest infinite covering show that equality holds in infinitely many cases.

To prove (ii) it remains to show

$$\frac{2}{3\sqrt{3}}A(Q_i)+\frac{d}{2}|p(d_i)-p(d_{i+1})|\leq 1/2,\qquad i=1,\dots,n-1 \tag{15}$$

for long edges $[a,b]$.

The first case is trivial, as clearly $2d \le 1/2$. In the other case we may assume by lemma 13

$$|d_i - p(d_i)| = |d_{i+1} - p(d_{i+1})| = \alpha, \quad |p(d_i) - p(d_{i+1})| = 2\sqrt{1-\alpha^2}$$

for $i = 2, \ldots, n-2$ and we have to show

$$\frac{4}{3\sqrt{3}}\alpha\sqrt{1-\alpha^2} + d\sqrt{1-\alpha^2} \le 1/2 \quad \text{for } 0 \le \alpha \le 1.$$

This inequality is certainly satisfied by

$$d = \min\{\frac{1}{2\sqrt{1-\alpha^2}} - \frac{4}{3\sqrt{3}}\alpha \,|\, 0 \le \alpha \le 1\}. \tag{16}$$

By using elementary calculus we find that the minimum is attained at

$$\begin{aligned}\alpha_0 &= \frac{\sqrt{8 - \frac{3}{(-4+\sqrt{17})^{\frac{1}{3}}} + 3\left(-4+\sqrt{17}\right)^{\frac{1}{3}}}}{2^{\frac{3}{2}}} \\ &= 0.6578\ldots\end{aligned}$$

Insertion of this value in 16 shows inequalty 15. For $i = 1, d$ it is easily checked that inequality 15 holds with strict inequality.

While the proof shows that for (2) equality only holds for card$X = 1$, the following example shows that the value of d is best possible: Let $\beta_0 = 2\sqrt{1-\alpha_0^2}$. Now for natural n let $X = \{x^1, \ldots, x^n, x_a, x_b, x_c, x_d\}$ with $x_i = ((i-1)\beta_0, 0), i = 1, \ldots, n,\ x_a = (-\beta_0, \alpha_0),\ x_b = (-\beta_0, -\alpha_0),\ x_c = ((n+1)\beta_0, \alpha_0),\ x_d = ((n+1)\beta_0, -\alpha_0)$. If $P = \text{conv}X$ is dissected as in the proof of the theorem, we obtain $2(n-1)$ quadrangles Q_i such that equality holds in inequality 15 and 6 additional triangles. qed

Proof of Corollary 11: Let C_n be an arbitrary covering with 2-balls. Then

$$\begin{aligned}\Theta(B^2, C_n, \rho) = \frac{\pi n}{V((\text{conv}C_n) + \rho B^2)} &\ge \frac{\pi\left(\frac{2}{3\sqrt{3}}V_2(\text{conv}C_n) + \beta V_1(\text{conv}C_n) + 1\right)}{V_2(\text{conv}C_n) + 2\rho V_1(\text{conv}C_n) + \pi\rho^2} \\ &= \frac{2\pi}{3\sqrt{3}} \frac{V_2(\text{conv}(C_n) + \frac{3\sqrt{3}}{2}V_1(\text{conv}C_n) + \frac{3\sqrt{3}}{2}}{V_2(\text{conv}C_n) + 2\rho V_1(\text{conv}C_n) + \pi\rho^2} \\ &> \frac{2\pi}{3} \quad \text{fr} \quad \rho \le \frac{3\sqrt{3}}{4}\beta.\end{aligned}$$

In the same way the examples from Theorem 10 give the estimate for large ρ, such that in fact it is easy to give effective ϵ_n. qed

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